PLASMA OSCILLATIONS OF AN ELECTRON BEAM IN AN EXTERNAL ELECTRIC FIELD

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Many papers have been devoted to the problem of the interaction of beams of charged particles with a plasma (a detailed bibliography is given, for example, in [1]). Analysis of the dispersion equation shows that in the case of a sufficiently slow monoenergetic electron beam of low density, growing longitudinal waves are not excited in a system consisting of such a beam and a plasma [2–4].

The problem of the penetration of an external longitudinal electric field into a semiconfined plasma with an electron beam in the absence of instabilities in the system is studied (the boundary-value problem for growing waves was examined in [5]). This problem is, in a certain sense, an extension of the second part of L. D. Landau's well-known work [6] to include the case of a plasma with a beam. On the other hand, in the absence of an external electric field, this problem may be considered a boundary-value problem of the interaction of a weakly modulated electron beam with a plasma.

1. Derivation of the integral equation. Let a plasma be confined by a plane wall that is an ideal reflector of particles incident on it, and let an electron beam with charge density ρ_0 and velocity v_0 relative to the plasma be propagated perpendicular to this plane into the interior of the plasma. It is assumed that there is no thermal velocity spread in the beam. Let the x axis lie along the wall in the direction of propagation of the beam and let u be the velocity component along this axis.

The distribution function $f(u, x)$ must have the property $f(u, 0) = f(-u, 0)$ at the boundary; in this case we shall use a distribution function integrated with respect to V_y and V_z.

The strength of the longitudinal electric field E_1 and the perturbations of the density ρ_1 and velocity v_1 of the beam are also specified at the boundary.

If the deviations from equilibrium are small, then the plasma oscillations of the system are described by linear equations [3]

$$-i\omega f + u\frac{\partial f}{\partial x} - \frac{e}{m}E\frac{df_0}{du} = 0,$$
$$-i\omega v + v_0\frac{dv}{dx} = -\frac{e}{m}E,$$
$$\frac{dE}{dx} = -4\pi e\int_{-\infty}^{\infty} f\,du + 4\pi\rho, \qquad (1.1)$$
$$-i\omega\rho + \rho_0\frac{dv}{dx} + v_0\frac{d\rho}{dx} = 0.$$

Here f is the deviation of the distribution function from a Maxwellian distribution function f_0; ρ the deviation of the charge density of the beam from the equilibrium value ρ_1, which is assumed to be compensated by an excess positive charge in the plasma; and v is the deviation of the beam velocity from the equilibrium value v_0. The dependence of all the quantities upon time is taken in the form $\exp(-i\omega t)$.

System of equations (1.1) can be reduced to an integral equation in E(x). Accordingly, formal integration of each equation of the system should be carried out beforehand.

Thus, from the last two equations of (1.1) we find

$$v = \left(v_1 - \frac{e}{mv_0}\int_0^x E\chi(-\xi)\,d\xi\right)\chi(x) \quad \left(\chi(\xi) = \exp\frac{i\omega\xi}{v_0}\right), \qquad (1.2)$$

$$\rho = \left\{\rho_1 - \frac{\rho_0}{v_0^2}\left[i\omega v_1 x - \frac{e}{m}\int_0^x E\chi(-\xi)\,d\xi - \frac{i\omega e}{mv_0}\int_0^x E\chi(-\xi)(x-\xi)\,d\xi\right]\right\}\chi(x). \qquad (1.3)$$

The relation linking f and E, which follows from the first equation of (1.1), is not included here; it coincides with the expression in L. D. Landau's article [6] and is necessary only for deriving the integral equation. Finally, integration of the second equation of (1.1) gives

$$E = E_1 + \frac{4\pi i}{\omega}\left(e\int_{-\infty}^{\infty} uf(u, x)\,du - \rho_0 v - \rho v_0 + j_1\right),$$
$$(j_1 = \rho_0 v_1 + \rho_1 v_0). \qquad (1.4)$$

The following relation [from the first equation of (1.1)] was used here:

$$i\omega\int_{-\infty}^{\infty} f\,du = \frac{d}{dx}\int_{-\infty}^{\infty} uf\,du.$$

Thus, we arrive at the following integral equation:

$$E(x) - \int_0^x L(x-\xi)E(\xi)\,d\xi - \int_0^x K(x-\xi)E(\xi)\,d\xi - \int_x^{\infty} K(\xi - x)E(\xi)\,d\xi + \int_0^{\infty} K(\xi + x)E(\xi)\,d\xi = \psi(x),$$

$$K(\xi) = \frac{4\pi i e^2}{m\omega}\int_0^{\infty}\frac{df_0}{du}\exp\frac{i\omega\xi}{u}\,du \qquad (\xi > 0),$$

$$L(\xi) = \frac{4\pi e\rho_0}{mv_0^2}\,\xi\exp\frac{i\omega\xi}{v_0} \qquad (\xi > 0),$$

$$\psi(x) = E_1 - \frac{mv_0v_1}{e}L(x) + \frac{4\pi i j_1}{\omega}\left(1 - \exp\frac{i\omega x}{v_0}\right). \qquad (1.5)$$

Note that the function $K(\xi)$ was investigated by L. D. Landau [6].

2. Integral representation of the solution. Let us put the integral equation in a form that is more convenient for solution. For example, the field E(x) is conveniently represented as the sum of two terms

$$E(x) = E_\infty + E^\circ(x). \qquad (2.1)$$

It is not difficult to show that the field strength when $x \to \infty$

$$E_\infty = \frac{E_1 + 4\pi i j_1/\omega}{1 - (\omega_-/\omega)^2 - (\omega_+/\omega)^2} = \frac{E_1 + 4\pi i j_1/\omega}{\varepsilon}, \qquad (2.2)$$

where ω_- and ω_+ are the Langmuir frequencies of the plasma without the beam and of the electron beam, respectively, and ε is the dielectric constant of the plasma without the beam. Let us formally extend the definitions of functions $K(\xi)$ and $L(\xi)$ and the unknown function $E^\circ(x)$ into the domain of negative values of the independent variable:

$$K(-\xi) = K(\xi), \qquad L(-\xi) = L(\xi),$$
$$E^\circ(-x) = -E^\circ(x). \qquad (2.3)$$

Then the integral equation for $E^\circ(x)$ can be written as

$$E^\circ(x) - \int_{-\infty}^{\infty} K(x-\xi)E^\circ(\xi)\,d\xi \mp \int_0^x L(x-\xi)E^\circ(\xi)\,d\xi =$$
$$= \pm g(\pm x), \qquad (2.4)$$

$$g(x) = \psi(x) - E_\infty + E_\infty \int_0^x [L(\xi) + 2K(\xi)]\,d\xi. \qquad (2.5)$$

Here the upper signs are for $x > 0$ and lower signs are for $x < 0$.

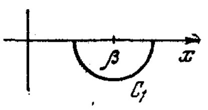

Fig. 1

Let us solve integral Eq. (2.4) by the Fourier method. If we multiply both sides of the equation by $\exp(-ikx)$ and integrate with respect to x from $-\infty$ to $+\infty$, we obtain

$$E^\circ(k)[1 - K(k)] - E_k^\circ L_k + E_{-k}^\circ L_{-k} = g_k - g_{-k}. \qquad (2.6)$$

Here, for any value of $\varphi(x)$, the symbols $\varphi(k)$ and φ_k are defined by the equations

$$\varphi(k) = \int_{-\infty}^{\infty} e^{-ikx}\varphi(x)\,dx,$$
$$\varphi_k = \int_0^{\infty} e^{-ikx}\varphi(x)\,dx. \qquad (2.7)$$

It is easy to see that if $\varphi(x)$ is an even function, then its Fourier component $\varphi(k) = \varphi_k + \varphi_{-k}$; but if $\varphi(x)$ is odd, then $\varphi(k) = \varphi_k - \varphi_{-k}$.

Assuming that it is odd, we represent Eq. (2.6) as

$$E_k^\circ[1 - L_k - K(k)] - E_{-k}^\circ[1 - L_{-k} - K(k)] =$$
$$= g_k - g_{-k}. \qquad (2.8)$$

In order to solve Eq. (2.8), we must establish a relationship between the functions of the independent variable -k and the complex-conjugate functions of the independent variable k, which is possible if the real and imaginary parts in $E^\circ(x)$, $g(x)$, $L(x)$, and $K(x)$ are separated and the transforms corresponding to then are examined separately. Then (2.8) is reduced to a system of two equations, which connect the imaginary parts of certain analytic functions of k. The real parts of these analytic functions can differ only by constants. By analyzing the behavior of the functions when $|k| \to \infty$, however, it is easy to show that the constants are equal to zero. As a result, we obtain

$$E^\circ_k = \frac{g_k}{1 - L_k - K(k)},$$
$$E^\circ(k) = \frac{g_k}{1 - L_k - K(k)} - \frac{g_{-k}}{1 - L_{-k} - K(k)}. \qquad (2.9)$$

Thus, the electric field is represented as

$$E = E_\infty +$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{g_k}{1 - L_k - K(k)} - \frac{g_{-k}}{1 - L_{-k} - K(k)}\right]e^{ikx}\,dk. \qquad (2.10)$$

3. Distribution of electric field and of small perturbations of beam velocity and density. Following [2, 6], let us introduce the functions $K_1(k)$ and $K_2(k)$, which are given by the relations

$$K_1(k) = \left(\frac{\omega_-}{\omega}\right)^2 \beta^2 [J_+(\beta) - 1],$$
$$K_2(k) = \left(\frac{\omega_-}{\omega}\right)^2 \beta^2 [J_-(\beta) - 1], \qquad (3.1)$$

where [2]

$$J_+(\beta) = \frac{\beta}{\sqrt{2\pi}}\int_{C_1} \exp\left(\frac{-x^2}{2}\right)\frac{dx}{\beta - x}$$
$$\left(\beta = \frac{\omega}{ku^\circ}, \quad u^\circ = \frac{\sqrt{\theta}}{\sqrt{m}}\right). \qquad (3.2)$$

Here θ is the temperature in energy units; m the electron mass; and the contour C_1 is shown in the figure.

The function $J_-(\beta)$ differs from $J_+(\beta)$ in that in integration the pole is circumvented not from below but from above. Therefore,

$$J_-(\beta) = J_+(\beta) + i\sqrt{2\pi}\,\beta\exp(-{}^1/_2\beta^2) \qquad (3.3)$$

It is easy to see that $K(k) = K_1(k)$ when $k > 0$ and $K(k) = K_2(k)$ when $k < 0$. Similarly, let us introduce the function $\Pi(k) = K_k - K_{-k}$, and also the functions $\Pi_1(k)$ and $\Pi_2(k)$, which are given by the following formulas:

$$\Pi_1(k) = \Pi_2(k) - \sqrt{2\pi}\, i\,\frac{\omega_-^2\,\omega}{k^3u^{\circ 3}}\exp\left(-\frac{\omega^2}{2k^2u^{\circ 2}}\right); \; \Pi_2(k) =$$
$$= -\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\omega_-^2}{\omega k u^\circ} + \frac{1}{\sqrt{2\pi}}\frac{\omega_-^2\omega}{k^3u^{\circ 3}}\exp\left(-\frac{\omega^2}{2k^2u^{\circ 2}}\right)\mathrm{Ei}\left(\frac{\omega^2}{2k^2u^{\circ 2}}\right) \qquad (3.4)$$

Here Ei(z) is an integral exponential function.

It can be shown that $\Pi(k) = \Pi_1(k)$ when $k > 0$ and $\Pi(k) = \Pi_2(k)$ when $k < 0$.

Taking (3.1) and (3.4) into account, from formula (2.10) we obtain

$$E(x) = E_\infty - \int_{-\infty}^{\infty} e^{ikx} \left[\frac{\Phi_+(k)}{1 - L_k - K_2(k)} + \frac{\Phi_-(k)}{1 - L_{-k} - K_2(k)} \right] dk -$$

$$- \int_0^{\infty} e^{ikx} [K_1(k) - K_2(k)] \left\{ \frac{F_+(k)}{[1 - L_k - K_1(k)][1 - L_k - K_2(k)]} + \right.$$

$$\left. + \frac{F_-(k)}{[1 - L_{-k} - K_1(k)][1 - L_{-k} - K_2(k)]} \right\} dk \qquad (3.5)$$

Here

$$\Phi_\pm = \frac{iE_\infty [L_{\pm k} - \Omega_+^2 + K_2(k) - \Omega_-^2 \pm \Pi_2(k)]}{2\pi k} +$$

$$+ \frac{2j_1}{\omega (k \mp \omega / v_0)} \pm \frac{m v_1 \omega_+^2}{2\pi e v_0 (k \mp \omega / v_0)^2},$$

$$F_\pm = \pm \frac{iE_\infty}{2\pi k} [1 - L_{\pm k} - K_2(k) + \Pi_2(k) \pm \varepsilon] +$$

$$+ \frac{2j_1}{\omega (k \mp \omega / v_0)} \pm \frac{m v_1 \omega_+^2}{2\pi e v_0 (k \mp \omega / v_0)^2},$$

$$L_k = \frac{\omega_+^2}{v_0^2 (k - \omega / v_0)^2}, \qquad \Omega_+ = \frac{\omega_+}{\omega}, \qquad \Omega_- = \frac{\omega_-}{\omega}. \qquad (3.6)$$

Integrals (3.5) can be completely calculated only numerically, but it is not difficult to obtain an asymptotic formula that gives the law of variation of the field E(x) at values of x that are great in comparison with the Debye radius a of the plasma without the beam. Using a calculation method completely analogous to that used in [6], we obtain the expression

$$E(x) = E_\infty \left\{ 1 + \frac{2}{\sqrt{3\varepsilon}} \Omega_-^{-4/3} \left(\frac{x}{a}\right)^{2/3} \exp\left[- \frac{3}{4} \left(\frac{x}{\Omega_- a}\right)^{2/3} \right] \times \right.$$
$$\left. \times \exp\left(i \left[\frac{3\sqrt{3}}{4} \left(\frac{x}{\Omega_- a}\right)^{2/3} + \frac{2\pi}{3} \right] \right) \right\} \qquad \left(a = \left(\frac{\theta}{4\pi n e^2}\right)^{1/2} \right). \qquad (3.7)$$

Expression (3.7) gives a law of decrease of the difference $E(x) - E_\infty$ that is similar to that in [6]. This was to be expected, since there is no thermal velocity spread in the beam. If $E_1 + 4\pi i j_1 / \omega$ vanishes, then the field E(x) will approach zero exponentially as $x \to \infty$, which follows from an evaluation of the integrals by determination of the residues.

If we know the law of distribution of the electric field in the plasma, then from (1.2) and (1.3) we can find the distribution of the perturbations of beam velocity and density. Let us convert these expressions to a form that is more convenient for deriving asymptotic formulas:

$$v(x) = \frac{e}{m v_0} \exp \frac{i\omega x}{v_0} \int_x^{\infty} E(\xi) \exp\left(- \frac{i\omega\xi}{v_0} \right) d\xi,$$

$$\rho(x) = - \frac{\rho_0}{v_0} v(x) + \qquad (3.8)$$

$$+ \frac{i\omega e \rho_0}{m v_0^3} \exp\left(\frac{i\omega x}{v_0}\right) \int_x^{\infty} E(\xi) \exp\left(- \frac{i\omega\xi}{v_0} \right) (\xi - x) \, d\xi.$$

For example, perturbation of the beam velocity at high values of x is connected with the electric field E(x) by the simple relation

$$v(x) = - \frac{ie}{m\omega} E(x). \qquad (3.9)$$

4. **Investigation of resonance.** Let us find the roots of the denominators of the integrands in (3.5); the dispersion equation $1 - L_k - K_2(k) = 0$ for longitudinal oscillations [2] can be represented as

$$1 - \left(\frac{\beta\Omega_+}{\beta - \nu}\right)^2 = \beta^2 \Omega_-^2 [J_-(\beta) - 1] \qquad \left(\nu = \frac{v_0}{u^\circ}\right). \qquad (4.1)$$

The poles with small Im k make a substantial contribution to integral (3.6). We shall therefore seek the roots of Eq. (4.1) that lie close to the essential singularity k = 0 in the upper half-plane k. Assuming that $|\beta| \gg 1$, expanding $1/(\beta - \nu)$ in powers of ν/β, and using the asymptotic form of the function $J_-(\beta)$ in the upper half-plane [2], we finally obtain

$$\frac{1}{\beta} = \frac{-\Omega_+^2 \nu \pm \sqrt{\Omega_+^4 \nu^2 + 3(\Omega_-^2 + \Omega_+^2 \nu^2)\varepsilon}}{3(\Omega_-^2 + \Omega_+^2 \nu^2)}. \qquad (4.2)$$

The roots of the equation $1 - L_{-k} - K_2(k) = 0$ are determined by the same formula (4.2), in which, however, ν changes sign. One of the roots (4.2) lies in the upper half-plane only when the radicand is negative, i.e.,

$$\varepsilon < - \frac{\Omega_+^4 \nu^2}{3(\Omega_-^2 + \Omega_+^2 \nu^2)}. \qquad (4.3)$$

Let us find the roots of the equation $1 - L_k - K_1(k) = 0$, which also lie in the upper half-plane k. This expression has the form

$$1 - \left(\frac{\beta\Omega_+}{\beta - \nu}\right)^2 = \beta^2 \Omega_-^2 [J_+(\beta) - 1]. \qquad (4.4)$$

The asymptotic form of $J_+(\beta)$ in the upper half-plane (which corresponds to the lower half-plane β) has an exponentially small imaginary term [2].

Let us represent the desired root of Eq. (4.4) as

$$\beta = \beta_0 (1 + \beta_1 / \beta_0). \qquad (4.5)$$

where β_0 is the real part of the root, which is determined from (4.2), ignoring the exponentially small term, and β_1 is a small imaginary component. Then

$$\frac{\beta_1}{\beta_0} = - \frac{i\Omega_-^2 \sqrt{\tfrac{1}{2}\pi}\, \beta_0^5 \exp(-\tfrac{1}{2}\beta_0^2)}{3\Omega_-^2 + \Omega_+^2 \nu (3\nu + \beta_0)}. \qquad (4.6)$$

From the statement of the problem, it follows that $\Omega_+ \ll \Omega_-$ and $\nu \ll 1$. Let us represent the dielectric constant ε as $\varepsilon = \varepsilon_* + \Delta\varepsilon$, where

$$\varepsilon_* = - \frac{\Omega_+^4 \nu^2}{3(\Omega_-^2 + \Omega_+^2 \nu^2)}. \qquad (4.7)$$

As we shall see below, the value of ε_* will be critical, since when $\varepsilon > \varepsilon_*$ the law of variation of the field near the boundary is qualitatively different as compared with the case $\varepsilon < \varepsilon_*$. In [6], the critical value was $\varepsilon_* = 0$. The shift of the critical value of ε_* into the domain of negative values of ε in the case in question is explained by the Doppler frequency reduction for a moving beam. Consider the cases $\Delta\varepsilon > 0$ and $\Delta\varepsilon < 0$. When $\Delta\varepsilon > 0$

$$\frac{1}{\beta_0} = \frac{-\Omega_+^2 \nu \pm \sqrt{3(\Omega_-^2 + \Omega_+^2 \nu^2)\Delta\varepsilon}}{3(\Omega_-^2 + \Omega_+^2 \nu^2)}. \qquad (4.8)$$

so that when $\nu > 0$ and $\Delta\varepsilon > -\varepsilon_*$, $\beta_0 > 0$ also, and there then exists a root of Eq. (4.4) with $\operatorname{Im}\beta_1 < 0$. When $\nu < 0$, there is always at least one such root of Eq. (4.4). Let $\Delta\varepsilon < 0$. Then in the lower half-plane β there is always a root of Eq. (4.1)

$$\frac{1}{\beta} = \frac{-\Omega_+^2\nu + i\sqrt{3(\Omega_-^2 + \Omega_+^2\nu^2)\,|\Delta\varepsilon|}}{3(\Omega_-^2 + \Omega_+^2\nu^2)}. \qquad (4.9)$$

Taking into account relations (4.2)–(4.9) and limiting the expansion in powers of k to linear terms, we find that when $\Delta\varepsilon < 0$ the electric field near the boundary varies according to the law

$$E = \frac{E_1}{\varepsilon}\left\{1 - 2\,\frac{\sqrt{2/\pi}\,\Omega_-^2 + \Omega_+^2\nu}{\Omega_+^2\nu}\times\right. \qquad (4.10)$$
$$\left.\times \exp\left[-\frac{x}{\Omega_- a}\left(\frac{|\Delta\varepsilon|}{3(\Omega_-^2 + \nu^2\Omega_+^2)}\right)^{1/2}\right]\cos\left[\frac{x}{\Omega_- a}\,\frac{\nu\Omega_+^2}{3(\Omega_-^2 + \nu^2\Omega_+^2)}\right]\right\}.$$

In order to shorten formula (4.10), we let $v_1 = 0$ and $j_1 = 0$.

Similarly, we obtain the behavior of the field near the boundary when $\Delta\varepsilon > 0$

$$E = \frac{E_1}{\varepsilon}\left\{1 + \frac{\varepsilon\gamma}{2\Omega_+^2\nu}\exp\left[\frac{ix}{\Omega_- a\gamma} - \frac{\Omega_- x}{\Omega_+ a}\,\frac{\sqrt{\pi}}{\sqrt{2}}\,\frac{\gamma^{5/2}}{\sqrt{\nu\Delta\varepsilon}}\exp\left(-\frac{\gamma^2}{2}\right)\right]\right\},$$
$$\gamma = \frac{3(\Omega_-^2 + \Omega_+^2\nu^2)}{\Omega_+^2\nu}. \qquad (4.11)$$

Formula (4.11) is also obtained assuming $v_1 = 0$ and $j_1 = 0$.

The necessity of taking into account the residue at the pole in evaluating the integral from zero to infinity, which leads to expression (4.11), is explained by the fact that, in finding the asymptotic form of this integral by the method of descent, the original contour of integration must be deformed so that it coincides with the level line that passes through the saddle point. In this case, the pole is circumvented in the right half of the upper half-plane, if it is situated below the level line and above the axis of abscissas.

In formula (4.11), $\Delta\varepsilon$ must be assumed to be small, but in this case γ must be great, so that the field slowly attenuates with increase in x. Otherwise, this term can be ignored, and the field can be determined from (3.7).

In the absence of a beam, Eqs. (4.10) and (4.11), which describe the resonance case, are inapplicable. But if the quadratic terms are taken into account in the expansion in powers of k, then passage to the limiting case of absence of a beam gives the relations obtained in [6].

At $x = 0$, formulas (4.10) and (4.11) do not give the correct boundary value of E_1, since terms of the order of $|\Delta\varepsilon|$ were ignored in the calculations. With increase in x, however, in both cases the field oscillates about the value E_1/ε, which it approaches at infinity.

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